## ON THE LIOUVILLE THREE-POINT FUNCTION

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The recently proposed expression for the general three point function of exponential fields in quantum Liouville theory on the sphere is considered. By exploiting locality or crossing symmetry in the case of those four-point functions, which may be expressed in terms of hypergeometric functions, a set of functional equations is found for the general three point function. It is shown that the expression proposed by the Zamolod-chikovs solves these functional equations and that under certain assumptions the solution is unique.

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Up to now exact information on exponential Liouville operators and their correlation functions was available only for a rather small subclass of these operators. On a heuristic level these results were based on the observation that the Liouville path integral with exponential operators inserted in some cases reduces to a path integral in a free field theory [GL]. These rather wild manipulations can be rigorously justified within the context of the exact operator quantization as developed by Gervais and collaborators [GN1][GN2][GS]. However, these methods only work in the case that the number of screening charges in a free field representation of operators and correlation functions is a positive integer. There have been attempts to extend these results to the case of a negative integer number of screening charges, based either on continuation prescriptions [GL][Do][Ki][FK] or on conjectures about the chiral algebra of the corresponding chiral vertex operators [Ge2], but rigorous justification for these proposals is still missing.

Recently a proposal was made for the exact general three point function of exponential Liouville field operators ([DO1][DO2] and independently in [ZZ]). If properly established, such a result could be an important progress beyond the known cases. It provides the information needed for the decomposition of arbitrary correlation functions into conformal blocks (for the general formalism see [BPZ], a proposal in the context of Liouville theory is eqn. (2.23) of [ZZ]), and should therefore be an important ingredient to study the factorization properties of Liouville theory. Factorization of Liouville correlation functions is a subtle issue [Sei][Pol], and one may expect important qualitative insights on the nature of Liouville theory and 2d gravity from such investigations.

The aim of the present note is to show that under certain additional assumptions the requirement of crossing symmetry [BPZ] may be used to give a proof for the expression proposed in [ZZ].

Let  $V_{\alpha}$  denote the Liouville field operator corresponding to the classical field  $e^{2\alpha\phi}$ . Application of these operators on the  $sl_2$ -invariant state<sup>1</sup>  $|0\rangle_L$  creates Virasoro highest weight states  $|\alpha\rangle_L$  with weight  $h_{\alpha} = \alpha(Q - \alpha)$ :

$$\lim_{z,\bar{z}\to\infty} V_{\alpha}(z,\bar{z})|0\rangle_{L} = |\alpha\rangle_{L}.$$
 (1)

To label Liouville states  $|\alpha\rangle$  by "momenta"  $\alpha$  is justified by the correspondence to free field states: The asymptotic behaviour of the wave-functional  $\Psi_{|\alpha\rangle}(\phi(z,\bar{z}))$  in the asymptotic limit when the zero-mode  $\phi_0$  goes to  $-\infty$  is given by the free field wave functions of highest weight states  $|\alpha\rangle_F$  with zero-mode momentum  $\alpha$ , see i.e. [Pol2] or [ZZ]. One gets the correspondence

$$|\alpha\rangle_L \sim |\alpha\rangle_F + R(\alpha)|Q - \alpha\rangle_F.$$
 (2)

For imaginary values of  $\alpha$ ,  $\alpha = iP + Q/2$ , one may interprete  $R(\alpha)$  as the amplitude for reflection on the Liouville wall.

The Liouville space of states is covered once by restricting  $\alpha - Q/2 \leq 0$ . However, it is convenient to also introduce states with  $\alpha - Q/2 \geq 0$ , defined by

$$|Q - \alpha\rangle_L = R^{-1}(\alpha)|\alpha\rangle_L \sim |Q - \alpha\rangle_F + R^{-1}(\alpha)|\alpha\rangle_F.$$
 (3)

Obviously, it is useful to define  $R(Q - \alpha) = R^{-1}(\alpha)$  in order to have the correspondence to free field states in the form of (2).

<sup>&</sup>lt;sup>1</sup>Existence of such a state has been questioned in the context of the Gervais-Neveu approach. However, in [T] it has been shown that most achievments of this approach are indeed compatible with the existence of an  $sl_2$ -invariant state.

From now on I will only consider Liouville states and drop the subscript L. The states dual to  $|\alpha\rangle$  will be denoted  $\langle\alpha|$ . The out-vacuum will be defined to be the  $sl_2$ -invariant state  $\langle Q|$ . I will assume that

$$\lim_{z,\bar{z}\to\infty} \langle Q|V_{\alpha}(z,\bar{z})|z|^{4\Delta_{\alpha}} = \langle Q-\alpha|$$
 (4)

This assumption identifies the reflection amplitude  $R(\alpha)$  with the two-point function  $\langle V_{\alpha}V_{\alpha}\rangle$ . The three point function is defined as

$$\mathcal{G}_{\alpha_3 \alpha_2 \alpha_1}(x_3, x_2, x_1) = \langle Q | V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_1}(z_1, \bar{z}_1) | 0 \rangle, \tag{5}$$

where  $x_i = (z_i, \bar{z}_i)$ .  $V_{\alpha}$  is supposed to be a primary conformal field with dimension  $\Delta_{\alpha} = \alpha(Q - \alpha)$ . The coordinate dependence is therefore determined to be [BPZ]

$$\mathcal{G}_{\alpha_3\alpha_2\alpha_1}(x_3, x_2, x_1) = |z_{12}|^{\Delta_{12}} |z_{13}|^{\Delta_{13}} |z_{23}|^{\Delta_{23}} C(\alpha_3, \alpha_2, \alpha_1), \tag{6}$$

where  $z_{ij} = z_i - z_j$ ,  $\Delta_{ij} = \Delta_k - \Delta_i - \Delta_j$  for  $k \neq i, j$  and  $\Delta_i \equiv \Delta_{\alpha_i}$ . The structure constants  $C(\alpha_3, \alpha_2, \alpha_1)$  are the main object of interest in the present note. As a consequence of locality, C has to be symmetric in its arguments.

A consequence of assumption (4) that will be important below is

$$\langle \alpha_3 | V_{\alpha_2}(z, \bar{z}) | \alpha_1 \rangle \propto C(Q - \alpha_3, \alpha_2, \alpha_1).$$
 (7)

Now consider the four point function of exponential Liouville operators, which will be written as

$$\mathcal{G}_{\alpha_4 \alpha_3 \alpha_2 \alpha_1}(x_4, x_3, x_2, x_1) = \langle Q | V_{\alpha_4}(z_4, \bar{z}_4) V_{\alpha_3}(z_3, \bar{z}_3) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_1}(z_1, \bar{z}_1) | 0 \rangle, \tag{8}$$

Projective invariance allows to reduce  $\mathcal{G}$  to a function of the cross-ratio

$$z = \frac{z_{21}z_{43}}{z_{31}z_{42}} \tag{9}$$

and its complex conjugate:

$$\mathcal{G}_{\alpha_4\alpha_3\alpha_2\alpha_1}(x_4, x_3, x_2, x_1) = (10)$$

$$= |z_{42}|^{-4\Delta_2} |z_{41}|^{2(\Delta_3 + \Delta_2 - \Delta_1 - \Delta_4)} |z_{43}|^{2(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)} |z_{31}|^{2(\Delta_4 - \Delta_1 - \Delta_2 - \Delta_3)} G_{\alpha_4\alpha_3\alpha_2\alpha_1}(z, \bar{z}).$$

As a consequence of (7) the decomposition of  $G_{\alpha_4\alpha_3\alpha_2\alpha_1}(z,\bar{z})$  into conformal blocks is of the general form

$$G_{\alpha_4\alpha_3\alpha_2\alpha_1}(z,\bar{z}) = \sum_{\alpha} C(\alpha_4,\alpha_3,\alpha)C(Q-\alpha,\alpha_2,\alpha_1) \left| \mathcal{F}_s \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}(z) \right|^2.$$
 (11)

An important property of  $\mathcal{G}_{\alpha_4\alpha_3\alpha_2\alpha_1}(x_4, x_3, x_2, x_1)$  that follows from the requirements of locality and associativity of the operator products is the so-called crossing-invariance [BPZ], which in terms of  $G_{\alpha_4\alpha_3\alpha_2\alpha_1}(z,\bar{z})$  is expressed as

$$G_{\alpha_4 \alpha_3 \alpha_2 \alpha_1}(z, \bar{z}) = G_{\alpha_4 \alpha_1 \alpha_2 \alpha_3}(1 - z, 1 - \bar{z}) = |z|^{-4\Delta_2} G_{\alpha_1 \alpha_3 \alpha_2 \alpha_4}(1/z, 1/\bar{z}).$$
 (12)

The strategy to get information on the coupling constants will be to exploit (12) in a special case where the conformal blocks in (11) can be determined explicitely:

For the special case that  $\alpha_2 = -b/2$  (resp.  $\alpha_2 = -1/2b$ ) it is well known [BPZ][GN2]<sup>2</sup> that the operator  $V_{\alpha_2}$  satisfies operator differential equations:

$$\frac{\partial^2}{\partial z^2} V_{\alpha_2}(z,\bar{z}) = b^2 (T_{<}(z) V_{\alpha_2}(z,\bar{z}) + V_{\alpha_2}(z,\bar{z}) T_{>}(z)),$$

$$T_{<}(z) = \sum_{n=-\infty}^{-2} z^{-n-2} L_n \qquad T_{>}(z) = \sum_{n=-1}^{\infty} z^{-n-2} L_n$$
(13)

and a corresponding equation for the antiholomorphic dependence. As a consequence, G satisfies the ordinary differential equations

$$\left(-\frac{1}{b^2}\frac{d^2}{dz^2} + \left(\frac{1}{z-1} + \frac{1}{z}\right)\frac{d}{dz} - \frac{\Delta_3}{(z-1)^2} - \frac{\Delta_1}{z^2} + \frac{\Delta_3 + \Delta_2 + \Delta_1 - \Delta_4}{z(z-1)}\right)G(z,\bar{z}) = 0.$$
(14)

It follows from (14) that the only values of  $\alpha$  that appear in (11) are  $\alpha_1 + sb/2$ ,  $s = \pm 1$  and that the conformal blocks  $\mathcal{F}_s \equiv \mathcal{F}_{\alpha_1 + sb/2}$  may be expressed in terms of the hypergeometric function F(A, B; C; z) as

$$\mathcal{F}_s(z) = z^{a_s} (1 - z)^b F(A_s, B_s; C_s; z)$$
(15)

where  $a_s = \Delta_{\alpha_1 + sb/2} - \Delta_2 - \Delta_1$ ,  $b = \Delta_{\alpha_3 - b/2} - \Delta_3 - \Delta_2$  and

$$A_s = -sb(\alpha_1 - Q/2) + b(\alpha_3 + \alpha_4 - b) - 1/2$$
(16)

$$B_s = -sb(\alpha_1 - Q/2) + b(\alpha_3 - \alpha_4) + 1/2 \tag{17}$$

$$C_s = 1 - sb(2\alpha_1 - Q). \tag{18}$$

The basic point here is that the identity

$$F(A, B; C; z) = \frac{\Gamma(C)\Gamma(B - A)}{\Gamma(B)\Gamma(C - A)} (-z)^{-A} F(A, 1 - C + A; 1 - B + A, 1/z)$$

$$\frac{\Gamma(C)\Gamma(A - B)}{\Gamma(A)\Gamma(C - B)} (-z)^{-B} F(B, 1 - C + B; 1 - A + B, 1/z). \tag{19}$$

yields a relation of the form

$$\mathcal{F}_{s}\begin{bmatrix} \alpha_{3} & \alpha_{2} \\ \alpha_{4} & \alpha_{1} \end{bmatrix}(z) = z^{-2\Delta_{2}} \sum_{t=+,-} B_{st} \, \mathcal{F}_{s}\begin{bmatrix} \alpha_{3} & \alpha_{2} \\ \alpha_{1} & \alpha_{4} \end{bmatrix}(1/z), \tag{20}$$

which may be used to exploit the crossing symmetry relations (12). One finds

$$\frac{C(\alpha_4, \alpha_3, \alpha_1 + b/2)}{C(\alpha_4, \alpha_3, \alpha_1 - b/2)} = -\frac{C_-(\alpha_1)}{C_+(\alpha_1)} \frac{B_{-+}\bar{B}_{--}}{B_{++}\bar{B}_{+-}},\tag{21}$$

where the notation  $C_s(\alpha) = C(\alpha, -b/2, Q - (\alpha + sb/2))$  has been used. This is the sought-for functional equation for  $C(\alpha_4, \alpha_3, \alpha_1)$ , since the right hand side may be determined explicitly: First, the explicit form of the  $B_{st}$  is found from (19). This yields

$$\frac{B_{-+}B_{--}}{B_{++}\bar{B}_{+-}} = -\frac{\gamma(b(2\alpha_1 - b))}{\gamma(2 - b(2\alpha_1 - b))} \times \frac{\gamma(b(-\alpha_1 + \alpha_3 + \alpha_4 - b/2))}{\gamma(b(\alpha_1 - b/2 + \alpha_3 + \alpha_4 - Q))\gamma(b(\alpha_1 - \alpha_3 + \alpha_4 - b/2))\gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))},$$
(22)

<sup>&</sup>lt;sup>2</sup>The relation between these rather different approaches is clarified in [T]

where  $\gamma(z) = \Gamma(z)/\Gamma(1-z)$  and the identity

$$\frac{\Gamma^2(2-z)}{\Gamma^2(z)} = -\frac{\gamma(2-z)}{\gamma(z)} \tag{23}$$

have been used.

Second, the  $C_s(\alpha)$  are among the coupling constants for which integral representations exist and have been computed in [DF], cf. also eqn. (3.7) of [ZZ].

$$\frac{C_{-}(\alpha_1)}{C_{+}(\alpha_1)} = -\frac{\gamma(-b^2)}{\pi\mu} \gamma(2\alpha_1 b) \gamma(2 - b(2\alpha_1 - b)). \tag{24}$$

Having found an functional equation for  $C(\alpha_4, \alpha_3, \alpha_1)$  one should compare it with the corresponding equation satisfied by the expression proposed in [ZZ], eqn. (3.14). Using the functional equation for the Upsilon-function,

$$\Upsilon(x+b) = \gamma(bx)b^{1-2bx}\Upsilon(x),\tag{25}$$

one finds

$$\frac{C(\alpha_3, \alpha_2, \alpha_1 + b)}{C(\alpha_3, \alpha_2, \alpha_1)} = -\frac{\gamma(-b^2)}{\pi \mu} \frac{\gamma(b(2\alpha_1 + b))\gamma(2b\alpha_1)\gamma(b(\alpha_2 + \alpha_3 - \alpha_1 - b))}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))\gamma(b(\alpha_1 + \alpha_2 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))},$$
(26)

which is equivalent to (21) with (23),(24).

One gets a second functional equation by replacing  $b \to b^{-1}$ . The solution to these functional equations is unique up to a constant factor if one assumes that b is irrational and that the dependence of C on its last argument is continuous: If there was a second solution  $D(\alpha_3, \alpha_2, \alpha_1)$  of the same functional equations, then R = D/C would have to be periodic with two incommensurable periods. It is amusing to note that it was J. Liouville, who first proved that such a function can only be constant, see [JL], chap. XIII. If one expands R in a Fourier-series over trigonometric functions with period b, then every term in the sum has to be  $b^{-1}$ -periodic seperately, which is possible only if R = const. Note that one needs only square-integrability of R in [0, b] to reach this conclusion.

A number of points deserve further discussion:

- 1. The condition of irrationality of b does not hold in the case of the minimal models and the corresponding  $c \geq 25$  Liouville theories needed for their coupling to gravity. The corresponding arbitrariness in the solution for C may be fixed by demanding something like continuous dependence on the parameter b. Furthermore, if  $1 \leq c \leq 25$  then the argument for uniqueness breaks down completely since b becomes complex. It is intriguing to see how the so-called (c = 1)-barrier appears in this context.
- 2. A closely related reasoning may be used in an operator approach such as that of Gervais and collaborators: First, the notations are related by

$$b \equiv \sqrt{\frac{h}{\pi}} \qquad \alpha - Q/2 \equiv -\frac{1}{2}\sqrt{\frac{h}{\pi}}\varpi. \tag{27}$$

Second, the four point functions to be considered read

$$\langle \varpi_4 | e^{-jb\phi}(1,1)e^{-\frac{1}{2}b\phi}(z,\bar{z}) | \varpi_1 \rangle,$$
 (28)

where

$$e^{-\frac{1}{2}b\phi}(z,\bar{z}) = \sum_{i=1,2} C_i(\varpi)V_i(z)\bar{V}_i(\bar{z}),$$
 (29)

the chiral components  $V_i(z)$  being characterized by the operator differential equation

$$\frac{d^2}{dz^2}V_i(z) = \frac{h}{\pi}(T_{<}(z)V_i(z) + V_i(z)T_{>}(z)). \tag{30}$$

Now the argument is somewhat dependent on the form that one assumes for the operator  $e^{-jb\phi}(1,1)$ : Consider the general ansatz

$$e^{-jb\phi}(z,\bar{z}) = \int dr \ C_r^j(\varpi) V_r^j(z) \bar{V}_r^j(\bar{z}), \tag{31}$$

where  $V_r^j(z)$  shifts the momentum by  $\varpi \to \varpi + 2j - 2r$ , such that r may formally be identified with the number of screening charges. The structure constants  $C_r^j(\varpi)$  correspond to  $C(Q-(\alpha-jb+rb),-jb,\alpha)$  in my previous notation. The requirement of mutual locality of  $e^{-jb\phi}(1,1)$  and  $e^{-\frac{1}{2}b\phi}(z,\bar{z})$  leads to the same functional equation as above, which now is of the form

$$\frac{C_r^j(\varpi_1 - 1)}{C_{r+1}^j(\varpi_1 + 1)} = -\frac{C_-(\varpi_1)}{C_+(\varpi_1)} \frac{B_{-+}\bar{B}_{--}}{B_{++}\bar{B}_{+-}},\tag{32}$$

If one assumes  $C_r^j(\varpi)$  to be a square-integrable function of  $r, j, \varpi$  then the previous argument for the uniqueness of the solution of the functional equation goes through. However, if the sum in the ansatz (31) for  $e^{-jb\phi}(z,\bar{z})$  includes only a discrete number of values of r then there is no reason to assume that  $C_r^j(\varpi)$  depends square-integrably on r. Nevertheless, if a value  $r_0$  appears in the sum, then the sum will also have to include a discrete series  $\{r_0+n\} \subset r_0+\mathbb{Z}$ , which is restricted only by possible zeroes or poles of the rhs of (32). Equation (32) may then be used for a recursive determination of  $C_{r_0+n}^j(\varpi)$  in terms of  $C_{r_0}^j(\varpi)$ . For  $r_0=0$  this will reproduce the results of [DF] on the structure constants as well as the Goulian-Li continuation prescription.

3. A puzzling issue is the following: In [Sei][Pol][ZZ] it is proposed that the sum over intermediate states mainly includes macroscopic states (hyperbolic sector), perhaps with a discrete sum of contributions of microscopic states (elliptic sector). However, in the present case  $\alpha_2 = -b/2$  one has only microscopic intermediate states. This is probably related to a point noted in [ZZ]: If poles cross the contour of integration in eqn. (2.23) of [ZZ] then one will get additional discrete contributions. Maybe in some cases the integration contour can be closed to yield only a finite sum of residues, which can be identified with contributions of microscopic states.

If the proposal of [Sei][Pol][ZZ] is right, then the ansatz for a general exponential Liouville operator would have to read

$$e^{-jb\phi}(z,\bar{z}) = \sum_{r} C_r^j(\varpi) V_r^j(z) \bar{V}_r^j(\bar{z}) + \int dP \ C^j(P,\varpi) V^j(P,z) \bar{V}^j(P,\bar{z}), \tag{33}$$

where  $V^{j}(P,z)$  maps a state from the elliptic sector into the hyperbolic sector. An understanding of these issues will be a major advance in Liouville theory.

## References

- [DO1] H. Dorn, H.J. Otto: On correlation functions for noncritical strings with  $C \le 1$ ,  $D \ge 1$ , Phys. Lett. **B291** (1992) 39
- [DO2] H. Dorn, H.J. Otto: Two and three point functions in Liouville theory, Nucl. Phys. **B429** (1994) 375
- [ZZ] A.B. Zamolodchikov, Al.B. Zamolodchikov: Structure constants and conformal bootstrap in Liouville field theory, preprint LPM-95/24, RU-95-39, hep-th/9506136
- [GN1] J.L. Gervais, A. Neveu: New quantum treatment of Liouville field theory, *Nucl. Phys.* **B224** (1982) 329-348
- [GN2] J.L. Gervais, A. Neveu: Novel triangle relation and absence of tachyons in Liouville string field theory, Nucl. Phys. B238 (1982) 125
- [Ge1] J.L. Gervais: The quantum group structure of 2D gravity and the minimal models I, Comm. Math. Phys. 130 (1990) 257-283
- [Ge2] J.L. Gervais: Gravity-matter couplings from Liouville theory, Nucl. Phys. **B391** (1993) 287-332
- [GS] J.L. Gervais, J. Schnittger: Continuous spins in 2D gravity: Chiral vertex operators and local fields, Nucl. Phys. B431 (1994) 273-314
- [T] J. Teschner: On quantization of Liouville theory and related conformal field theories, Ph. D. thesis, preprint DESY 95-118
- [Sei] N. Seiberg: Notes on Liouville theory and quantum gravity, *Progr. Theor. Phys. Suppl.* **102** (1990) 319-349
- [Pol] J. Polchinski: Remarks on Liouville field theory, in "Strings 90", eds. R. Arnowitt et al., World Scientific 1991
- [Pol2] J. Polchinski: Ward identities in two-dimensional gravity, Nucl. Phys. **B357** (1991) 241-270
- [GL] M. Goulian, M. Li: Correlation functions in Liouville theory, Phys. Rev. Lett. 66 (1991) 2051
- [Do] Vl. S. Dotsenko: Three-point correlation functions of the minimal conformal theories coupled to 2D gravity, Mod. Phys. Lett. 6 (1991) 3601
- [Ki] Y. Kitazawa: Gravitational descendents in Liouville theory, Phys. Lett. B265 (1991) 262
- [FK] P. Di Francesco, D. Kutasov: World-sheet and space-time physics in two-dimensional (super)string theory, Nucl. Phys. **B375** (1992) 119-170
- [BPZ] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov: Infinite conformal symmetry in 2D quantum field theory, *Nucl. Phys.* **B241** (1984) 333

- [DF] V.S. Dotsenko, V.A. Fateev: Conformal algebra and multipoint correlation functions in two-dimensional statistical models, Nucl. Phys. **B240** (1984) 312, and: Four point correlation functions and the operator algebra in two-dimensional conformal invariant theories with the central charge C < 1, Nucl. Phys. **B251** (1985) 691
- [JL] J. Lützen: Joseph Liouville, 1809-1882: Master of pure and applied mathematics, Springer 1990